Structural Synthesis Capability for Integrally Stiffened Waffle Plates

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Structural synthesis has been defined as the rational directed evolution of a structural configuration, which in terms of a defined criterion, efficiently performs a set of specified functional purposes. An automated, minimum weight, optimum design capability for rectangular, simply supported waffle plates subjected to a multiplicity of inplane load conditions is presented. The synthesis involves six design parameters (total depth, back-up sheet thickness, stiffener thickness in the x and y directions, and stiffener spacing in the x and y directions). The behavior constraints are provided primarily by elastic stability failure modes. In particular, gross instability of the waffle plate, buckling of the back up sheet panel, and stiffener buckling are prevented in each of several load conditions while seeking a minimum weight design configuration. Side constraints in the form of upper and lower bounds on the design parameters due to external considerations (such as space limits and manufacturing capabilities) are provided. The mathematical programming problem to which this structural synthesis problem leads has the following characteristics: 1) relative minima; 2) nonlinear inequality constraints; 3) a multitude of side constraints; and 4) nonlinear merit function. Numerical results for several symmetric as well as unsymmetrical waffle-plate optimum designs are presented. The results of the waffle-plate synthesis study support the contention that structural synthesis capabilities for complex structural configurations of current and future importance are feasible.

Nomenclature

A_{mn}	= participation coefficient of assumed mode
a	= x dimension of plate
b	= y dimension of plate
b_x	= x spacing of stiffeners
b_y	= y spacing of stiffeners
b_{Lx}, b_{Ly}	= lower bound on b_x, lower bound on b_y
$(b_x)_{\max}, (b_y)_{\max}$	= upper bound on b_x, upper bound on b_y
C_a	= shear buckling coefficient
D_1	= bending stiffness, x direction
D_2	= bending stiffness, y direction
D_3	= torsional stiffness
D_p	= pth design parameter
E	= modulus of elasticity
H	= total height of stiffener plus skin
N_x	= intensity of resultant normal force, x direction
N_{xy}	= intensity of resultant shear force, x and y directions
N_y	= intensity of resultant normal force, y direction
R_i	= ith random number
t_{wx}	= thickness of stiffeners parallel to x axis
t_{wy}	= thickness of stiffeners parallel to y axis
t_s	= sheet thickness
$(t_s)_{\min}$	= lower bound on t_s
$(t_{wx})_{\min}, (t_{wy})_{\min}$	= lower bound on t_{wx}, lower bound on t_{wy}
w	= z component of displacement
W	= total weight of a waffleplate
Y	= yield stress
α_r	= aspect ratio, a/b
μ	= Poisson's ratio
ρ	= weight density

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Introduction

TRADITIONALLY the task of engineering has been to determine the best possible design of a system which is to perform a useful function. Early structural engineers were confronted with the problem of designing a stress limited determinate structure subjected to a single critical load condition. Since no external side constraints were placed on the independent variables of design, the well-known theorems of Maxwell[1] and Michell,[2] concerned with the minimization of structural material, were natural developments. H. L. Cox[3, 4] furthered their work through the development of practical applications. Schmidt[5] applies these notions to the design of redundant trusses. However, the fully stressed design criteria is not generally equivalent to weight minization except for stress limited statically determinate trusses (see Ref. 6, p. 131).

Catchpole[7] applied the ideas of the substitute problem of full utilization to the design of structural members susceptible to a buckling failure. Employing the vehicle of a simple Euler column and designing against both gross failure and local crippling, several investigators have concluded that the minimum weight optimum design must lie at the intersecton of constraints. In other words it is assumed at the outset that the minimum weight optimum design has the property that all failure modes are equally likely to occur under the single critical load condition, and then a design having this property is sought. It must be emphasized that necessary conditions for this method to successfully achieve an optimum design are 1) the existence of only as many possible true failures modes as there are design parameters; and 2) unconstrained design parameters, free to assume any real value independent of physical implications. It must be recognized that these are necessary but not sufficient conditions for being able to find a design with the property that all failure modes are equally likely under the single critical load condition.

Later investigators have developed design tables to aid the designer in the selection of many types of structural members subject to various load conditions. References 8, 9, and 10 are only a few examples of this type of investigation. When using this type of approach, one must bear in mind not only the foregoing restrictions but also the following: 1) the tables are developed for single load condition situations; and 2) the design generated is a collection of individual member optima, and structural interaction between these member optima is not considered.

Gerard[11] developed a slightly modified approach for waffle plates but maintained the basic assumption of simultaneous failure modes which carries the foregoing restrictions. Structural synthesis as presented herein was developed with the goal of overcoming these restrictions by returning to the fundamental problem of weight minimization.

Methods of structural analysis which adequately predict the behavior of a substantial class of structural systems exist. The wide spread availability of large, high-speed, digital computers has made it possible to carry out analyses of complex structures rapidly, and thus bring the results to bear on the design. It has become possible to complete several design cycles; however, for the most part, the redesign process has been executed by engineers using the results obtained with automated analysis capabilities. Usually the redesign process is not explicitly delineated, but rather it is a subtle and complex combination of physical insight, experience, judgment, and often courage. The desire to exploit more fully the evergrowing body of experimental and analytical capability for predicting structural behavior, in terms of superior designs, has led rather naturally to a quest for automated methods of optimum design.

Investigators from the fields of economics, management, and mathematics have, during the past twenty years, developed the methodology of operations research, which has come to be called mathematical programming.[12-15] There is a growing awareness that a substantial class of engineering design problems may be viewed as mathematical programming problems. Klein[16] observed that the minimum weight design of structures can be formulated as a classical extremum problem using Lagrangian multipliers and slack variables to introduce inequality constraints. Pearson[17] has given a method of automated minimum weight design based on the limit analysis philosophy. Hilton[18] discusses the problem of synthesizing a minimum weight structural design that will have a prescribed probability of survival. Kalaba[19] develops the dynamic programming approach to Hilton's work. Dennis[20] discusses the general mathematical programming problem in the abstract and shows that determination of the direction of constrained steepest descent is, in itself, a quadratic programming problem. It should be noted that the method of constrained steepest descent converges toward the relative minimum, which, in general, is not necessarily the global (absolute) minimum. Feder,[21] in a most entertaining presentation, discusses the lens design problem within the framework of mathematical programming.

In Ref. 6 structural synthesis was defined as the rational directed evolution of a structural design which, in terms of a defined criterion, efficiently performs a set of specified functional purposes. Structural synthesis is essentially a problem in mathematical programming involving three types of considerations; namely, a specified set of requirements, a given technology, and a criterion by means of which choices can be made between various designs.

Requirements

The basic requirement of the structural system is that it must maintain its structural integrity while being subject to the design load system. The design is inadequate and the structure is said to fail if the structural behavior does not remain within the confines of the stated limits. What constitutes failure must be carefully defined and this can be expected to vary from one design task to another.

In addition to the behavioral requirements, limitations on the design parameters describing the structural system also must be satisfied. These requirements are called side constraints and arise for reasons such as analysis limitations, compatability constraints, and fabrication limitations.

Technology

The method of analysis used to predict the behavior of proposed designs is a prerequisite to development of a structural synthesis capability and will be referred to as the governing technology. Existing literature contains methods of analysis which adequately predict the behavior of a substantial class of structural systems. It should be noted that there has been a tendency to regard a problem as being solved when a reliable method of analysis has been found, but, in fact, availability of a governing technology is only a point of departure for tackling the structural synthesis task.

Criterion

In many areas of structural design the minimization of weight is important. It is noteworthy that a minimum weight basis for evaluating merit is probably the most readily stated. This is fortuitous in view of the importance of minimum weight design in space vehicle structures. Minimization of weight is treated as the sole criterion of merit in this paper. However, it should be pointed out that the structural synthesis concept may be applied using another criterion, for example, total cost, provided it is expressible mathematically.

In Ref. 6 a general structural synthesis problem for simple discrete elastic systems was formulated and the three bar truss was used as an example illustrating the concept. In Ref. 22 the three bar truss synthesis capability was applied to material and configuration selection on a discrete variable basis, and in Ref. 23 the influence of including elementary elastic stability constraints was reported. In Ref. 24 the three bar truss synthesis capability was extended so that it was possible to automatically size the members, select the configuration, and determine the combination of attainable material properties in order to achieve a minimum weight optimum design. The program reported in Ref. 24 illustrates the feasibility of extending synthesis capabilities upward into the design parameter hierarchy, provides an efficient means for material selection from existing materials, suggests that structural synthesis can provide a means of determining attainable material properties that are desirable for a particular job based on structural optimization, and offers some background in dealing with a nine dimensional design parameter space.

In order to examine the contention that structural synthesis capabilities can be developed for complex structural systems of current and future importance it was decided to focus attention on a structural system where the design parameters describe the actual configuration rather than an idealized structure, and in which consideration of both local and over-all instability constraints was essential. The integrally stiffened waffle-plate structure shown in Fig. 1,

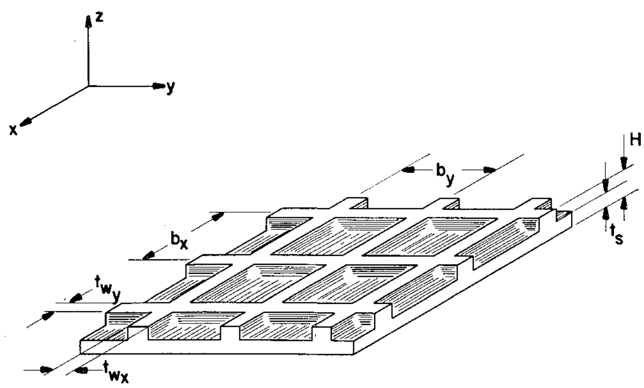

Fig. 1 Integrally stiffened waffle-like plate.

subject to inplane load conditions, was selected as a structural system exhibiting both of the previously stated characteristics. This paper reports on the development of a structural synthesis capability for waffle plates which represents a considerable advance beyond the exploratory truss studies. The successful completion of the waffle-plate synthesis program adds to the growing body of evidence supporting the contention that a structural synthesis capability can be developed for complex structural systems of practical importance. Before formulating the waffle-plate synthesis problem it will be useful to examine certain fundamental concepts and definitions employed in structural synthesis.

Fundamentals of Structural Synthesis

At the outset, certain parameters in any design problem are fixed. All those parameters which are not preassigned are called design parameters $\{D_p\}$. These independent variables are to be determined by the synthesis program such that the merit function assumes the optimal value. Consider an n dimensional orthogonal space in which there is a coordinate axis for each design parameter. This space will be referred to as a design parameter space. The coordinates of any point in the positive region of the space identify a unique design of the system. Assuming the merit function and its first partial derivatives with respect to the design parameters are continuous, the gradient to the merit function is unique for each point in the design space.

Within the design parameter space there exists behavior constraints and side constraints. If a design point is on a behavior constraint surface, the structure is on the verge of failure in one of the defined modes. If a design point is on a side constraint surface, the structure is on the verge of being unacceptable with respect to the design parameter limitations imposed on the system. A design point which lies on a constraint (within a tolerance ϵ) is said to be a bound point; all other points in the design space are said to be free points. Associated with each defined failure mode, there exists a behavior constraint surface for each load condition of the design load system. The collection of behavioral and side constraint surfaces which separates the acceptable region§ of the design space from the unacceptable region is called the composite constraint surface.‖ The composite constraint surface is usually continuous, but the gradient to the composite constraint surface is usually discontinuous at the junction of any two contributing surfaces. Any point in the design parameter space may be identified as either (see Fig. 2) 1) free and acceptable, 2) bound and acceptable, 3) free and unacceptable, or 4) bound and unacceptable.

The behavior of the structure is tested or examined through the mechanism of a behavior function. A behavior function is a mathematical expression relating the coordinates of the proposed design and the design requirements to the behavior of the structure. If the ith behavior function $BF_{ij}(D_p)$ is within its given limits for the jth load condition, then the design under examination does not fail in the ith failure mode under the jth load condition. If the following relationship holds

$$L_{ij} \leq BF_{ij}(D_p) \leq U_{ij} \qquad \begin{matrix}\text{for all } i \\ \text{for all } j\end{matrix} \quad (1)$$

§ The acceptable region may be composed of several disjoint sets. It might in some instances contain one point, in which case the problem is to find that point. The acceptable set may be a null set in which case the problem has no solution.

‖ In the case of n design parameters the composite constraint surface is the totality of points satisfying some $F(D_p) = 0$ which is a subspace $C_n - 1$ and divides the design space into two subsets; those points for which $F(D_p) < 0$ and those points for which $F(D_p) > 0$. In this sense it is a surface; it is not a two dimensional object but $n - 1$ space, which may or may not be disjoint.

where L_{ij} and U_{ij} are the upper and lower limits, the design is said to be acceptable with respect to the behavior constraints. It should be noted that limits on each failure mode in each load condition can be assigned independently.

The limitations on the design parameters may be expressed mathematically as follows:

$$D_p^{(L)} \leq D_p \leq D_p^{(U)} \qquad \text{for } p = 1 \rightarrow n \quad (2)$$

When this relationship is satisfied, the design is said to be acceptable with respect to the side constraints. It should be noted that the upper $D_p^{(U)}$ and lower $D_p^{(L)}$ limits on each design parameter may be given as constants or functions of the other design parameters.

The total weight of the structure can be expressed as a function of the design parameters and is denoted by $W(D_p)$. The minimization of total structural weight is taken as the sole criterion of merit in this paper.

Using matrix notation, a rather general class of structural synthesis problems now may be stated concisely as follows:

Given the preassigned parameters and the design load system as well as the design requirements $\{D_p^{(U)}\}$, $\{D_p^{(L)}\}$,

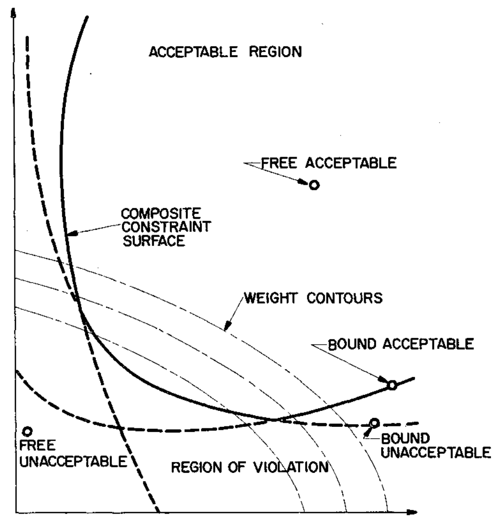

Fig. 2 Design parameter space nomenclature.

$[L_{ij}]$, and $[U_{ij}]$; find $\{D_p\}$ subject to $\{D_p^{(L)}\} \leq \{D_p\} \leq \{D_p^{(U)}\}$# and $[L_{ij}] \leq [BF_{ij}(D_p)] \leq [U_{ij}]$ such that $W(D_p)$ takes on a minimum value.

It will be assumed that it is possible to establish an initial trial design that is acceptable without recourse to the structural synthesis capability. Successful modification of the design can be accomplished by moving in the design parameter space, and insisting that the total weight of the structure does not increase. For simplicity the redesign steps are restricted to straight lines, that is,

$$\{D_p^{(q+1)}\} = \{D_p^{(q)}\} + \{\psi_p\}t \quad (3)$$

where $\{\psi_p\}$ is the column matrix of direction cosines defining a straight line of travel, t is the distance of travel, and the q superscript is the design cycle counter.

A major aspect of the development of methods of structural synthesis is the selection of proper directions and distances of travel in design parameter space. The following is a list of some of the available methods.

1) Directions
   a) random—employ a random number generator to develop the direction cosines
   b) directed—steepest descent
   c) semidirected—orient a line in the design parameter space emanating from this current design

It is to be understood that the matrix inequalities must be satisfied element by element.

to a predetermined point, e.g., point of equal weight or zero weight
2) Distances
   a) arbitrary—fixed increment or a random increment
   b) accelerated—a wide variety of methods for increasing and decreasing the increment size so as to more rapidly reach a bound point or a point at which the merit is no longer improved can be devised
   c) exact—solve for the distance to a neighboring constraint surface or a point on the same weight contour

Various schemes of design modification can be generated by combining the just mentioned directions and distances of travel.

Synthesis techniques that have been used so far have evolved on a rather pragmatic basis. Methods of redesign which, through repeated application, lead to an optimum design have been regarded as adequate. Comparative evaluation of alternate techniques constitutes a second stage optimization problem wherein the best method of finding the optimum design is sought. Improved methods of automated redesign and comparative evaluation of existing techniques require further study. It is clear that improvements in synthesis technique serve to extend the range of design problems for which an automated optimum design capability is attainable within the framework of any fixed digital computer capacity.

Formulation of the Waffle-Plate Problem

Consider a simply supported, rectangular waffle plate with integral orthogonal stiffeners subject to membrane loading (see Figs. 1 and 3). The over-all dimensions a and b are preassigned and it is assumed that the material to be used is selected in advance. A single load condition is specified by giving N_x, N_y, and N_{xy} (see Fig. 3). The design load system is made up of several distinct load conditions. The existing digital computer program has an upper limit of five independent load conditions.

The development of a waffle-plate synthesis capability was carried out in two phases. Initially, the number of independent design parameters was limited to three by preassigning the total depth of the waffle plate H and admitting only symmetric stiffener arrangements, i.e., $b_x = b_y$ and $t_{wy} = t_{wx}$. The design parameters for the phase I program are the thickness of the sheet t_s, the stiffener spacing b_x, and the stiffener thickness t_{wy}. The column matrix locating any design point in the three dimensional design parameter space is defined as follows:

$$\{D_p\} = \begin{Bmatrix} t_s \\ b_x \\ t_{wy} \end{Bmatrix} \tag{4}$$

The three types of side constraints considered in this formulation are 1) fabrication limitations, e.g., lower bounds on sheet and stiffener thicknesses, 2) analysis limitations, e.g., upper bound on stiffener spacing, and 3) compatibility constraints, e.g., stiffener spacing must be greater than the stiffener thickness. The side constraints employed in the phase I waffle-plate synthesis program are specified in Eqs. (5) and (6). The numbers to the right of each element indicate the type of side constraint involved:

$$\{D_p^{(U)}\} = \begin{Bmatrix} \mathrm{H} \\ (b_x)_{\max} \\ b_x \end{Bmatrix} \begin{matrix} 3) \\ 2) \\ 3) \end{matrix} \tag{5}$$

$$\{D_p^{(L)}\} = \begin{Bmatrix} (t_s)_{\min} \\ b_{Lx} \\ (t_{wy})_{\min} \end{Bmatrix} \begin{matrix} 1) \\ 1) \text{ or } 3) \\ 1) \end{matrix} \tag{6}$$

In Eqs. (5) and (6), $(b_x)_{\max}$ is the maximum stiffener spacing consistent with an equivalent plate analysis and b_{Lx} is the larger of $(b_x)_{\min}$ and t_{wy}. Note that $(t_s)_{\min}$, $(t_{wy})_{\min}$, and $(b_x)_{\min}$ are all minimum dimensions to be assigned based on fabrication considerations.

The waffle plate is to be designed so that gross buckling, local buckling, and yielding do not occur in any of the given load conditions. In developing the behavior functions (see Appendix A) it is assumed that the structural material used exhibits an ideal elastic-plastic behavior. The waffle plate will be considered acceptable if 1) yielding under the plane stress state in the sheet does not occur; 2) yielding under the uniaxial stress state in the stiffeners parallel to the x axis does not occur; 3) yielding under the uniaxial stress state in the stiffeners parallel to the y axis does not occur; 4) gross plate buckling, that is elastic instability of the equivalent orthotropic plate under combined loading (N_x, N_y, N_{xy}), does not occur; 5) lateral buckling of the stiffeners parallel to the x axis does not occur; 6) lateral buckling of the stiffeners paralle lto the y axis does not occur; and 7) plate buckling of the sheet does not occur.

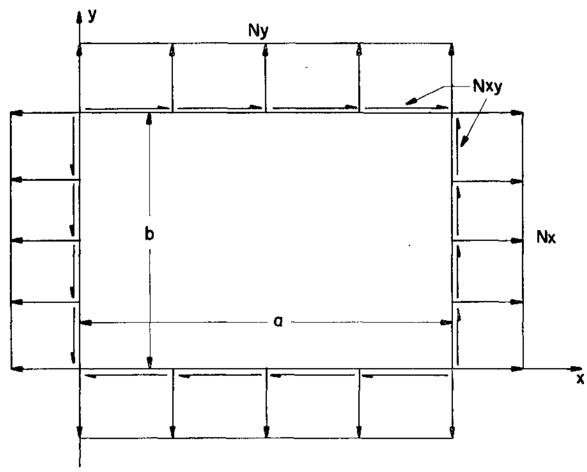

Fig. 3 Applied load sign convention.

It was found that (see Appendix A) all the behavior functions have the following characteristics: 1) they may be expressed so that the upper limit is unity and there is no lower bound, and 2) they are nonlinear functions of the design parameters. The column matrix of behavior functions for a single load condition can now be expressed as follows:

$$\{BF\ (D_p)\} = \begin{Bmatrix} GY\ (D_p) \\ SX\ (D_p) \\ SY\ (D_p) \\ GBF\ (D_p) \\ LBX\ (D_p) \\ LBY\ (D_p) \\ LBP\ (D_p) \end{Bmatrix} \tag{7}$$

where the elements of the behavior function matrix represent the behavior functions associated with the following failure modes:

$GY\ (D_p)$ = gross yield, Eq. (A19)
$SX\ (D_p)$ = stiffener yield, x direction Eq. (A20)
$SY\ (D_p)$ = stiffener yield, y direction Eq. (A21)
$GBF\ (D_p)$ = gross plate buckling, Eq. (A3)
$LBX\ (D_p)$ = local stiffener buckling, x direction Eq. (A11)
$LBY\ (D_p)$ = local stiffener buckling, y direction Eq. (A12)
$LBP\ (D_p)$ = local sheet buckling, Eq. (A14)

and the equation numbers refer to appropriate expressions in Appendix A which contains the governing technology.

The total weight for the phase I symmetric waffle plate may be expressed as follows:

$$W(D_p) = ab\rho H\left[1 - \left(1 - \frac{t_s}{H}\right)\left(1 - \frac{t_{wy}}{b_x}\right)^2\right] \quad (8)$$

where t_s, b_x, and t_{wy} are the design parameters and a, b, ρ, and H are previously defined preassigned parameters.

In the phase II waffle-plate synthesis the total depth H is not preassigned and the restriction to symmetric stiffener arrangements is removed. There are then six independent design parameters as shown in Eq. (4′):

$$\{D_p\} = \begin{Bmatrix} t_s \\ H \\ b_x \\ b_y \\ t_{wx} \\ t_{wy} \end{Bmatrix} \quad (4')$$

The side constraints employed are specified in Eqs. (5′) and (6′):

$$\{D_p^{(U)}\} = \begin{Bmatrix} H \\ (H)_{max} \\ (b_x)_{max} \\ (b_y)_{max} \\ b_y \\ b_x \end{Bmatrix} \quad (5')$$

$$\{D_p^{(L)}\} = \begin{Bmatrix} (t_s)_{min} \\ t_s \\ b_{Lx} \\ b_{Ly} \\ (t_{wx})_{min} \\ (t_{wy})_{min} \end{Bmatrix} \quad (6')$$

In Eq. (5′), $(b_x)_{max}$, and $(b_y)_{max}$ are the maximum stiffener spacings consistent with an equivalent plate analysis, and $(H)_{max}$ is upper limit on the total depth of the waffle plate. In Eq. (6′), b_{Lx} is the larger of $(b_x)_{min}$ and t_{wy}, whereas b_{Ly} is the larger of $(b_y)_{min}$ and t_{wx}. Note that $(t_s)_{min}$, $(t_{wy})_{min}$, $(t_{wx})_{min}$, $(b_x)_{min}$, and $(b_y)_{min}$ are all minimum dimensions to be assigned based on fabrication considerations. The column matrix of behavior functions for a single load condition can be expressed as follows:

$$\{BF(D_p)\} = \begin{Bmatrix} GY'(D_p) \\ SX'(D_p) \\ SY'(D_p) \\ GBF'(D_p) \\ LBX'(D_p) \\ LBY'(D_p) \\ LBP'(D_p) \end{Bmatrix} \quad (7')$$

where the elements of the behavior function matrix represent the behavior functions associated with the following failure modes:

$GY'(D_p)$ = gross yield (A19′)
$SX'(D_p)$ = stiffener yield, x direction (A20′)
$SY'(D_p)$ = stiffener yield, y direction (A21′)
$GBF'(D_p)$ = gross plate buckling (A3)
$LBX'(D_p)$ = local stiffener buckling, x direction (A11′)
$LBY'(D_p)$ = local stiffener buckling, y direction (A12′)
$LBP'(D_p)$ = local sheet buckling (A14)

and the equation numbers refer to appropriate expressions in Appendix A which contains the governing technology.

The total weight for the phase II waffle plate may be expressed as follows:

$$W = \rho abH\left[1 - \left(1 - \frac{t_s}{H}\right)\left(1 - \frac{t_{wx}}{b_y}\right)\left(1 - \frac{t_{wy}}{b_x}\right)\right] \quad (8')$$

where t_s, H, b_x, b_y, t_{wx}, and t_{wy} are the design parameters and a, b, and ρ are preassigned parameters.

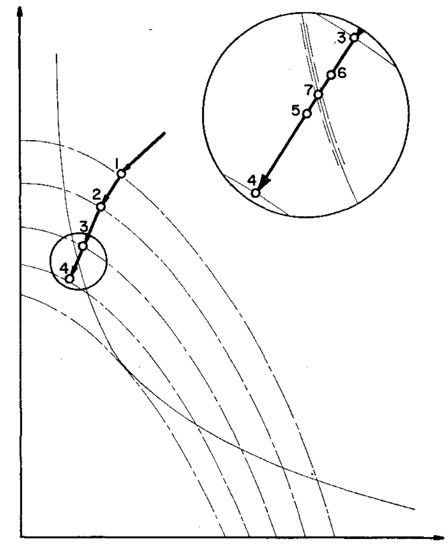

Fig. 4 Phase I: steepest descent hypothetical design space.

It is apparent that both the phase I and the phase II waffle-plate synthesis problems fall within the general class of structural synthesis problems stated in the previous section of this paper.

Synthesis Technique

The general approach to be followed in modifying the design of the structure was discussed previously. In this section the particular synthesis procedures used by the phase I and phase II waffle-plate synthesis program will be described. Both methods are of the alternate step type, that is, the method of steepest descent is used in moving from a free point whereas an alternate step mode of operation, in which the weight is held constant, is used in moving from a bound point.

It is assumed that a conservative initial trial design can be selected and given to the program as data. The preassigned parameters for the phase I program are the overall dimensions a and b, the total depth of the waffle H, and the waffle plate material properties E, Y, ρ, and μ. Input data for a particular phase I synthesis problem is completed by giving the side constraint limits $(t_s)_{min}$, $(b_x)_{min}$, $(b_x)_{max}$, and $(t_{wy})_{min}$ as well as the loading system matrix $[N]$ which, in its most general form, appears as follows:

$$[N] = \begin{bmatrix} N_x^{(1)} & N_x^{(2)} & N_x^{(3)} & N_x^{(4)} & N_x^{(5)} \\ N_y^{(1)} & N_y^{(2)} & N_y^{(3)} & N_y^{(4)} & N_y^{(5)} \\ N_{xy}^{(1)} & N_{xy}^{(2)} & N_{xy}^{(3)} & N_{xy}^{(4)} & N_{xy}^{(5)} \end{bmatrix} \quad (9)$$

Starting from a free acceptable initial design, the program enters the steep descent mode of operation. The direction of travel in steepest descent is found by calculating the negative of the components of the gradient to the weight function at any trial design point. The direction cosines which make up the elements of the column matrix $\{\psi_p\}$ [see Eq. (3)] are

$$\psi_1 = -\frac{(\partial W/\partial t_s)}{L} \quad (10)$$

$$\psi_2 = -\frac{(\partial W/\partial b_x)}{L} \quad (11)$$

$$\psi_3 = -\frac{(\partial W/\partial t_{wy})}{L} \quad (12)$$

where

$$L = \left[\left(\frac{\partial W}{\partial t_s}\right)^2 + \left(\frac{\partial W}{\partial b_x}\right)^2 + \left(\frac{\partial W}{\partial t_{wy}}\right)^2\right]^{1/2} \quad (13)$$

$$\frac{\partial W}{\partial t_s} = \left(1 - \frac{t_{wy}}{b_x}\right)^2 \rho ab \quad (14)$$

$$\frac{\partial W}{\partial b_x} = -\frac{2t_{wy}}{b_x^2}\left(1 - \frac{t_s}{H}\right)\left(1 - \frac{t_{wy}}{b_x}\right)\rho abH \quad (15)$$

$$\frac{\partial W}{\partial t_{wy}} = \frac{2}{b_x}\left(1 - \frac{t_s}{H}\right)\left(1 - \frac{t_{wy}}{b_x}\right)\rho abH \quad (16)$$

An increment of $t = 0.01$ was used in the steep descent mode of travel. The $(q + 1)$th trial design is determined from the qth trial design as shown in Eq. (3), where $t = 0.01$ and the $\{\psi_p\}$ are given by Eqs. (10–16). Each new design is tested with respect to the side constraints [Eqs. (5) and (6)] and the behavior functions for each load condition [Eq. (7)]. If the new design is acceptable and free, the steep descent mode of operation is continued. If it is not acceptable, the distance of travel to the composite constraint surface is found by systematically halving the current distance of travel until the incremental step is less than $\frac{1}{100}$th of the initial step length and the current design is considered a bound acceptable point. The steep descent mode of operation is illustrated schematically in a hypothetical two-dimensional design parameter space in Fig. 4.

Whenever the current trial design is a bound acceptable design the alternate step mode of operation is employed. Essentially, a free point design is sought which has the same weight as the current bound design. The technique to be described generates proposed alternate step designs which have the same weight as the current bound design and at the same time essentially satisfy all of the side constraints with the possible exception of $(b_x)_{\min} \leq b_x$. A random number generator is employed to select a plane parallel to the t_s axis and containing the current bound design point. The equation of such a plane is

$$Bb_x + Ct_{wy} = D_c \quad (17)$$

where

$$B = R_1/(R_1^2 + R_2^2)^{1/2} \quad (17a)$$

$$C = R_2/(R_1^2 + R_2^2)^{1/2} \quad (17b)$$

$$R_i = \text{random number between } -1 \text{ and } +1$$

and

$$D_c = Bb_x^{(q)} + Ct_{wy}^{(q)} \quad (17c)$$

The random plane represented by Eq. (17) will intersect the b_x, t_{wy} plane in a line. This line of intersection will be classified as (see Fig. 5): type 1, when it intersects the $b_x \leq (b_x)_{\max}$ bound and the compatibility bound $b_x > t_{wy}$; type 2, when it intersects the b_x axis and the compatibility bound $b_x > t_{wy}$; type 3, when it intersects $b_x \leq (b_x)_{\max}$ bound and the b_x axis. The random plane containing the current bound design point can be identified as type 1, 2, or 3 as follows:

1) set $t_{wy} = 0$ and solve Eq. (17) for b_x; if $b_x < 0$ or $b_x > (b_x)_{\max}$ then the random plane examined is type 1.

2) set $b_x = 0$ and solve Eq. (17) for t_{wy}; if $t_{wy} > 0$ the random plane examined is type 2, and if $t_{wy} < 0$ the random plane examined is type 3.

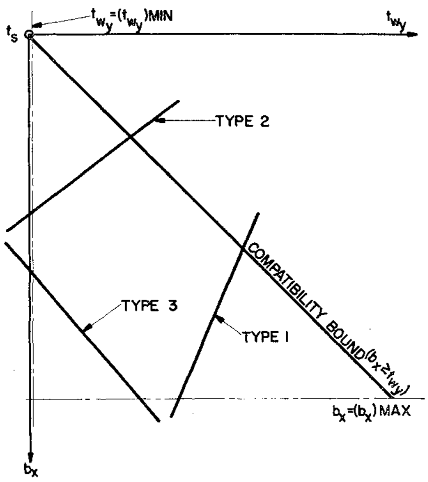

Fig. 5 Phase I: types of random planes.

The trace of the random plane through the design parameter space is depicted in Fig. 6. The plane of the paper is the random plane containing the current bound trial design point. The components of the gradient to the weight surface are given by Eqs. (14), (15), and (16), and are used to write an expression for the plane tangent to the weight surface at the current bound design point:

$$\left(\frac{\partial W}{\partial t_s}\right)_c t_s + \left(\frac{\partial W}{\partial b_x}\right)_c b_x + \left(\frac{\partial W}{\partial t_{wy}}\right)_c t_{wy} = K_c \quad (18)$$

where the subscript c indicates that the quantity is to be evaluated at the current bound design point. The intersection of the tangent plane with the random plane appears as a tangent line in Fig. 6. For each type of random plane it is possible to solve for the coordinates of the points defined

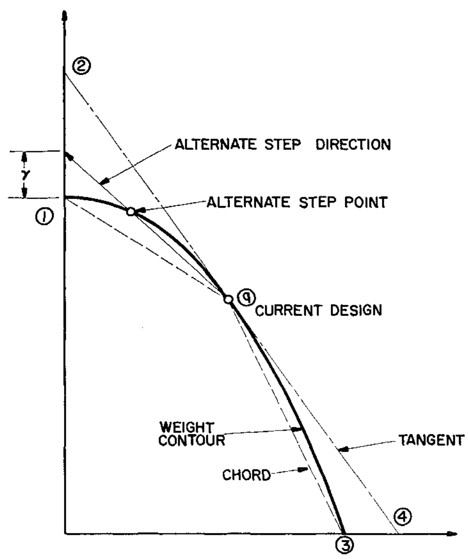

Fig. 6 Phase I: alternate step.

by the intersection of 1) the random plane, the tangent plane to the weight surface, and the appropriate design parameter bound; and 2) the random plane, the current weight surface, and the appropriate design parameter bound.

In Fig. 6 the tangent intersections are labeled 2 and 4 while the weight contour intersections are labeled 1 and 3. The current design point is labeled (q). The coordinates of points 1, 2, 3, and 4 for each type of random plane (1, 2, 3) are given in Appendix B. Let the vectors **a**, **b**, **c**, and **d** be the position vectors locating points 1, 2, 3, and 4 with respect to the origin. And let **q** be the position vector locating the current bound design point. For a particular random plane the components of the vectors **a**, **b**, **c**, and **d** can be determined and the alternate step direction of travel is found as follows:

$$\mathbf{r}_1 = (\mathbf{a} - \mathbf{q}) + \gamma(\mathbf{b} - \mathbf{a}) \quad (19)$$

or

$$\mathbf{r}_2 = (\mathbf{c} - \mathbf{q}) + \gamma(\mathbf{d} - \mathbf{c}) \quad (20)$$

where γ is a fraction less than unity. A unit vector in the alternate step direction is given by

$$\boldsymbol{\psi}_1 = \frac{\mathbf{r}_1}{|\mathbf{r}_1|} \quad (21)$$

or

$$\boldsymbol{\psi}_2 = \frac{\mathbf{r}_2}{|\mathbf{r}_2|} \tag{22}$$

It is possible to examine several points along the weight contour between 1 and q, or 3 and (q) by using different γ values. The following sequence of γ values is used by the phase I program:

0.5000	0.8750
0.7500	0.1250
0.2500	0.0625
0.6250	0.9375
0.3750	0.03125

The sequence is arranged so that the alternate step direction will tend to direct the path toward the center of the acceptable region. If the current design point is in a large pocket, the tendency is for the alternate step to remain in that same pocket. On the other hand, if the design point is in a small pocket, it is possible for the synthesis path to leave that pocket if a more dominant relative minimum exists.

The components of the unit vectors $\boldsymbol{\psi}_1$ or $\boldsymbol{\psi}_2$ are the elements of the design parameter modification matrix $\{\psi_p\}$ [see Eq. (3)] when in alternate step mode. The distance of travel t is determined by requiring that $W^{(q)} = W^{(q+1)}$ which may be written as

$$\left(1 - \frac{t_s^{(q)}}{H}\right)\left(1 - \frac{t_{wy}^{(q)}}{b_x^{(q)}}\right)^2 = \left[1 - \frac{(t_s^{(q)} + \psi_1 t)}{H}\right] \times \left[1 - \frac{(t_{wy}^{(q)} + \psi_3 t)}{(b_x^{(q)} + \psi_2 t)}\right]^2 \tag{23}$$

and solved for t. Any real value of t which is a root of Eq. (23) is used to generate a proposed alternate step design as shown in Eq. (3). If the proposed alternate step design is a free acceptable design, the program returns to the steep descent mode of operation until a constraint surface is encountered again. The synthesis technique for the phase I program is discussed in greater detail in Ref. 25, and a complete listing of the program will be found in Ref. 26. The computer program was written in the Runcible compiler language for the Burroughs 220 Computer.

The preassigned parameters for the phase II program are the over-all dimensions a and b, and the waffle plate material properties E, Y, ρ, and μ. Again it is assumed that a conservative initial trial design can be selected and given to the program as data. Input data for a particular phase II synthesis problem is completed by giving the side constraint limits $(H)_{\max}$, $(b_x)_{\max}$, $(b_y)_{\max}$, $(t_s)_{\min}$, $(b_x)_{\min}$, $(b_y)_{\min}$, $(t_{wx})_{\min}$, and $(t_{wy})_{\min}$ as well as the loading system matrix $[N]$ [see Eq. (9)].

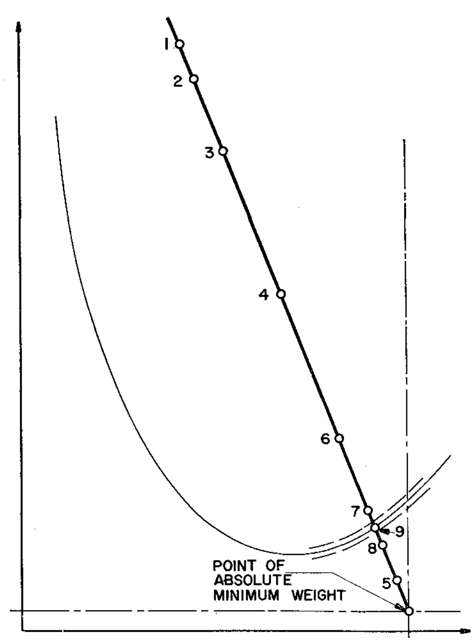

Fig. 7 Phase II: initial straight shot accelerated incrementation.

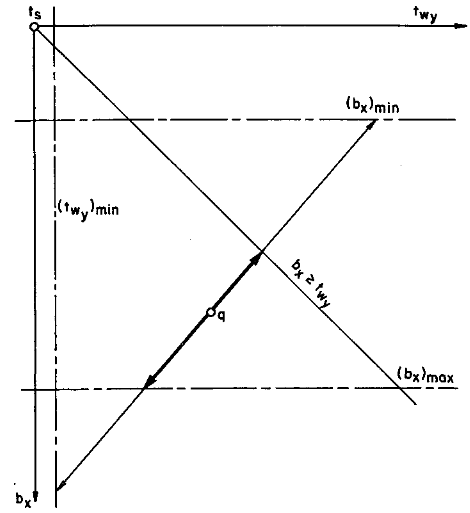

Fig. 8 Simplified illustration of alternate base planes method.

Starting from a free acceptable initial trial design, the direction of travel passing through the absolute minimum weight design point, satisfying only the side constraints, is used. The components of $\{\psi_p\}$ for this direction of travel are

$$\psi_1 = \frac{(t_s)_{\min} - t_s^{(1)}}{\tilde{L}} \qquad \psi_2 = \frac{(t_s)_{\min} - H^{(1)}}{\tilde{L}}$$
$$\psi_3 = \frac{(b_x)_{\max} - b_x^{(1)}}{\tilde{L}} \qquad \psi_4 = \frac{(b_y)_{\max} - b_y^{(1)}}{\tilde{L}} \tag{24}$$
$$\psi_5 = \frac{(t_{wx})_{\min} - t_{wx}^{(1)}}{\tilde{L}} \qquad \psi_6 = \frac{(t_{wy})_{\min} - t_{wy}^{(1)}}{\tilde{L}}$$

where

$$\tilde{L} = \{[(t_s)_{\min} - t_s^{(1)}]^2 + [(t_s)_{\min} - H^{(1)}]^2 + [(b_x)_{\max} - b_x^{(1)}]^2 + [(b_y)_{\max} - b_y^{(1)}]^2 + [(t_{wx})_{\min} - t_{wx}^{(1)}]^2 + [(t_{wy})_{\min} - t_{wy}^{(1)}]^2\}^{1/2} \tag{25}$$

Starting with an initial increment $t = \delta_0$ where δ_0 is given data (either 0.01 or 0.005), the (q + 1)th trial design is determined from the qth trial design as shown in Eq. (3), and the elements of $\{\psi_p\}$ are given by Eq. (24). Each new design is tested with respect to the side constraints [Eqs. (5′) and (6′)] and the behavior functions for each load condition [Eq. (7′)]. If the new design is acceptable and free, it becomes the current trial design and the increment size is doubled. Accelerated travel is continued along the direction given by Eq. (24) until a design which is unacceptable is found. Once a violation occurs, the distance of travel to the composite constraint surface is found by systematically halving the current distance of travel until the design point is on the constraint with a tolerance ϵ** or the increment size becomes so small as to represent an insignificant change in any design parameter. The last acceptable design point is

** If any $BF_{ij}\,(D_p) = 1.000 \pm 0.001$, the design point is considered bound. If, through the scheme of halving the current distance of travel $t \leq \delta_0$, $\delta_0/10$, $\delta_0/100$, $\delta_0/10{,}000$ (this option is controlled through the console by the operator) then the design point is considered bound even if $BF_{ij}\,(D_p) = 1.000 \pm 0.001$ is not attained.

considered a bound point. The accelerated incrementation procedure employed is illustrated schematically in a hypothetical two-dimensional design parameter space in Fig. 7.

In the phase II program, whenever the current design is a bound acceptable design, a form of alternate step operation is used. The technique employed is called the method of alternate base planes. The $(q+1)$th trial design, when (q) is a bound design, is determined as shown in Eq. (3), where five of the six elements in $\{\psi_p\}$ are given by

$$\psi_p = R_p \Bigg/ \left(\sum_{i=1}^{5} R_i^2\right)^{1/2}$$

and the remaining element is initially considered to be zero. The element initially set to zero is subsequently determined so that the proposed alternate step design has the same weight as the current bound design $\{D_p^{(q)}\}$. The element considered zero is selected by incrementing through the repeated sequence 1, 2, 3, 4, 5, 6; 1, 2, 3, 4, 5, 6, . . . , etc. In order that the proposed alternate step designs satisfy most of the side constraints, the distances t to the side constraints are calculated from the following expressions††:

Maximum Depth

$$t = \frac{H_{\max} - H^{(q)}}{\psi_2} \qquad \text{if } \psi_2 \neq 0 \quad (26)$$

Maximum x Stiffener Spacing

$$t = \frac{(b_x)_{\max} - b_x^{(q)}}{\psi_3} \qquad \text{if } \psi_3 \neq 0 \quad (27)$$

Maximum y Stiffener Spacing

$$t = \frac{(b_y)_{\max} - b_y^{(q)}}{\psi_4} \qquad \text{if } \psi_4 \neq 0 \quad (28)$$

Minimum x Stiffener Spacing

$$t = \frac{b_x^{(q)} - (b_x)_{\min}}{-\psi_3} \qquad \text{if } \psi_3 \neq 0 \quad (29)$$

Compatability Constraint $b_x \geq t_{w_y}$

$$t = \frac{b_x^{(q)} - t_{w_y}^{(q)}}{\psi_6 - \psi_3} \qquad \text{if } \psi_3 \neq 0 \text{ and } \psi_6 \neq 0 \quad (30)$$

Minimum y Stiffener Spacing

$$t = \frac{b_y^{(q)} - (b_y)_{\min}}{-\psi_4} \qquad \text{if } \psi_4 \neq 0 \quad (31)$$

Compatability Constraint $b_y \geq t_{w_x}$

$$t = \frac{b_y^{(q)} - t_{w_x}^{(q)}}{\psi_5 - \psi_4} \qquad \text{if } \psi_4 \neq 0 \text{ and } \psi_5 \neq 0 \quad (32)$$

Minimum x Stiffener Width

$$t = \frac{t_{w_x}^{(q)} - (t_{w_x})_{\min}}{-\psi_5} \qquad \text{if } \psi_5 \neq 0 \quad (33)$$

†† Application of the alternate base planes approach to the phase I three-dimensional synthesis problem can be easily visualized. The alternate base planes approach was not used in the phase I program and the following note is intended only to illustrate the basic idea behind the alternate base plane method. In Fig. 8, the plane of the paper is a plane normal to the t_s axis containing the bound design point (q). A random direction in this plane through point (q) is shown and the distances to the side constraints $(t_{w_y\min}, t_{w_y} \leq b_x, (b_x)_{\max}$, and $(b_x)_{\min}$ can be seen. In the instances shown in Fig. 9, t_1 and t_2 are given by the $(b_x)_{\max}$ bound and the $t_{w_y} \leq b_x$ compatibility constraint.

Minimum y Stiffener Width

$$t = \frac{t_{w_y}^{(q)} - (t_{w_y})_{\min}}{-\psi_6} \qquad \text{if } \psi_6 \neq 0 \quad (34)$$

Minimum Sheet Thickness

$$t = \frac{t_s^{(q)} - (t_s)_{\min}}{-\psi_1} \qquad \text{if } \psi_1 \neq 0 \quad (35)$$

Compatability Constraint $H \geq t_s$

$$t = \frac{H^{(q)} - t_s^{(q)}}{\psi_1 - \psi_2} \qquad \text{if } \psi_1 \neq 0 \text{ and } \psi_2 \neq 0 \quad (36a)$$

$$t = \frac{(W^{(q)}/\rho ab) - t_s^{(q)}}{\psi_1} \qquad \text{if } \psi_2 = 0 \quad (36b)$$

$$t = \frac{H^{(q)} - (W^{(q)}/\rho ab)}{-\psi_2} \qquad \text{if } \psi_1 = 0 \quad (36c)$$

From the set of values t, obtained using Eqs. (26–36), the smallest positive value is selected and designated t_1 and the negative value having the smallest absolute value is selected and designated t_2. Three random numbers between 0 and

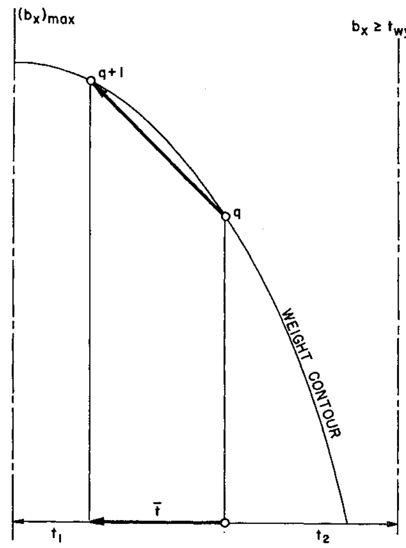

Fig. 9 Phase II: alternate step mode of travel.

1 are generated and multiplied times t_1 to give a distance of travel in the base plane designated $\bar{t}$. Taking each of these three values of $\bar{t}$ in sequence, the design parameter associated with the component of the $\{\psi_p\}$ vector initially set to zero is now evaluated so that the proposed alternate step design $\{D_p^{(q+1)}\}$ will have the same weight as the current bound design $\{D_p^{(q)}\}$ using the following expression:

$$W^{(q)} = \rho ab(H^{(q)} + \psi_2\bar{t}) \times \left\{1 - \left[\frac{(t_s^{(q)} + \psi_1\bar{t})}{(H^{(q)} + \psi_2\bar{t})}\right]\left[1 - \frac{(t_{w_x}^{(q)} + \psi_5\bar{t})}{(b_y^{(q)} + \psi_4\bar{t})}\right] \times \left[1 - \frac{(t_{w_y}^{(q)} + \psi_6\bar{t})}{(b_x^{(q)} + \psi_3\bar{t})}\right]\right\} \quad (37)$$

If the proposed alternate step design is a free acceptable design, the program enters the steep descent mode of operation. If not, the six proposed alternate step designs are examined one at a time. If none of the six are found to be acceptable and free, then the base plane is changed and a new set of proposed alternate step designs is generated. This process is continued until a free acceptable design is obtained

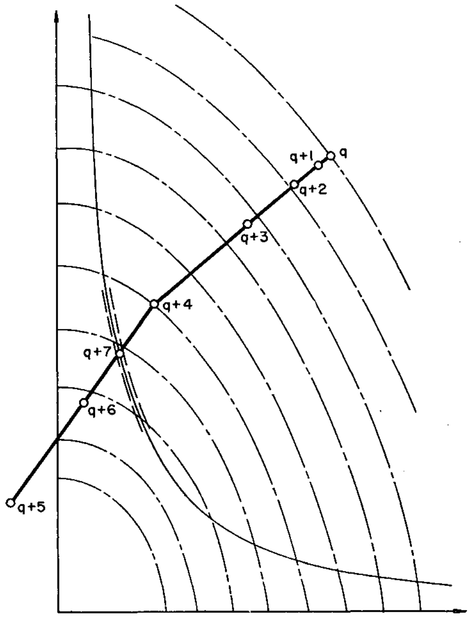

Fig. 10 Phase II: accelerated steepest descent.

or the current bound design is accepted as a proposed optimum design.‡‡

Except for the initial direction employed to reach the composite constraint surface the program always employs the steep descent mode of operation when the current trial design is an acceptable free point. The direction of travel in steep descent is found by calculating the negative of the components of the gradient to the weight function at any trial design point. The elements of the column matrix $\{\psi_p\}$ [see Eq. (3)] are

$$\{\psi_p\} = \frac{-(\partial W/\partial D_p)}{\left[\sum_{p=1}^{6}\left(\frac{\partial W}{\partial D_p}\right)^2\right]^{1/2}} \tag{38}$$

where

$$\frac{\partial W}{\partial t_s} = \rho ab\left[\left(1 - \frac{t_{wx}}{b_y}\right)\left(1 - \frac{t_{wy}}{b_x}\right)\right] \tag{39}$$

$$\frac{\partial W}{\partial H} = \rho ab\left[1 - \left(1 - \frac{t_{wx}}{b_y}\right)\left(1 - \frac{t_{wy}}{b_x}\right)\right] \tag{40}$$

$$\frac{\partial W}{\partial b_x} = -\frac{\rho abH}{b_x}\frac{t_{wy}}{b_x}\left(1 - \frac{t_s}{H}\right)\left(1 - \frac{t_{wx}}{b_y}\right) \tag{41}$$

$$\frac{\partial W}{\partial b_y} = -\frac{\rho abH}{b_y}\frac{t_{wx}}{b_y}\left(1 - \frac{t_s}{H}\right)\left(1 - \frac{t_{wy}}{b_x}\right) \tag{42}$$

$$\frac{\partial W}{\partial t_{wx}} = \frac{\rho abH}{b_y}\left(1 - \frac{t_s}{H}\right)\left(1 - \frac{t_{wy}}{b_x}\right) \tag{43}$$

$$\frac{\partial W}{\partial t_{wy}} = \frac{\rho abH}{b_x}\left(1 - \frac{t_s}{H}\right)\left(1 - \frac{t_{wx}}{b_y}\right) \tag{44}$$

Starting with an initial increment $t = \delta_0$, where δ_0 is given data (either 0.01 or 0.005) the $(q + 1)$th trial design is determined from the qth trial design as shown in Eq. (3), where the elements of $\{\psi_p\}$ are given by Eqs. (38–44). Each new design is tested with respect to the side constraints [Eqs. (5′) and (6′)] and the behavior functions for each load condition [Eq. (7′)]. If the new design is acceptable and free, it becomes the current trial design, the direction of steep descent is recalculated, and the increment size is doubled. Accelerated steep descent is continued until a design which is unacceptable is found. Once a violation occurs the distance of travel to the composite constraint surface is found by systematically halving the current distance of travel, until the design point is on the constraint within a tolerance ϵ, or the increment size becomes so small as to represent an insignificant change in any design parameter. The last acceptable design point is considered a bound design point. Note that during this process the direction of steep descent is re-evaluated at each new acceptable design. The phase II steep descent mode of operation is illustrated schematically in a hypothetical two-dimensional design parameter space in Fig. 10. Whenever the current trial design is a bound design point, the alternate step mode (phase II—alternate base planes procedure) of operation is employed to seek a free point. The phase II synthesis technique is discussed in greater detail in Ref. 27.

Both the phase I and phase II programs are capable of seeking the global minimum in a space containing several relative minima. The synthesis techniques described for phase I and phase II were developed with the relative minima problem in mind. Nevertheless, the vexing question as to whether or not the proposed optimum designs are indeed global minimum designs remains. A mathematical means of proving that a proposed optimum design is indeed a global or absolute minimum weight design is not known at present. However, there are a variety of ways to build a satisfactory degree of confidence in the results obtained. Both the phase I and phase II programs search a more or less uniform set of points in the weight contour of the current bound design point in a quest for a free point. The more designs tested, the higher the degree of confidence that the apparent optimum is the true optimum (assuming the acceptable region is not a disjoint set).

Another method of verifying the optimum, although it is not conclusive, is to run the synthesis from two distinct trial design points. If the synthesis paths are different but the final optimum is the same, within the same computation tolerance a degree of confidence can be achieved. Both of the foregoing procedures have been used to build confidence in the proposed minimum weight optimum designs presented herein. In the absence of a conclusive means of proving an optimum design, it is only possible to attain a reasonable degree of confidence with the verification procedures discussed.

Results and Discussion

In this section results from several examples demonstrating the waffle-plate synthesis capabilities developed are presented and discussed. Table 1 contains a summary of results obtained using the phase I symmetric waffle-plate synthesis program and Table 2 lists the material properties used.

Cases (1-1), (1-2), and (1-3) are single load condition problems using aluminum alloy as the plate material. They are identical except that the assigned total depth is different in each case (H = 0.4, 0.6, and 0.8 in.). In case (1-1) it is found that a thick sheet design (t_s = 0.2803) is the minimum weight optimum design. Note that the stiffener thickness is at the lower bound value (t_w = 0.0101 in.) and the only active behavior function is the gross buckling behavior function [GBF (D_p) = 0.9996].

In case (1-2) it is found that the minimum weight optimum design is characterized by a thin sheet with thick stiffeners. Gross buckling is the only active constraint in this case [GBF (D_p) = 1.0000]. In case (1-3) it is found that the minimum weight optimum design is characterized by thin sheet with relatively thin stiffeners. Both the gross buckling

‡‡ It should be noted that special modes of operation are automatically employed when on the 1) $t_{wy} = b_x$ compatibility bound, 2) $t_{wx} = b_y$ compatibility bound, and 3) $t_s = H$ compatibility bound. An optional externally controlled mode of operation, where t_{wy}/b_x and t_{wx}/b_y are held constant, is available.

Table 1 Summary of phase 1 results

Case	Preassigned parameters		Load conditions, N				Design parameters				Behavior functions, $BF(D_p)$		
			1	2	3		$D_p^{(L)}$	D_p	$D_p^{(U)}$		1	2	3
(1-1)	a = 40 in.	N_x	−0.30			t_s	0.005	0.2803	0.40	GBF	0.9996[a]		
	b = 30 in.	N_y	−0.40			b_x	2.00	4.3735	6.00	LBX	0.0364		
	H = 0.4 in.	N_{xy}	+0.20			t_{wy}	0.010	0.0101[a]	b_x	LBY	0.0485		
	aluminum					W		34.034		LBP	0.0157		
(1-2)	a = 40 in.	N_x	−0.30			t_s	0.005	0.0397	0.60	GBF	1.0000[a]		
	b = 30 in.	N_y	−0.40			b_x	2.00	2.5339	6.00	LBX	0.0040		
	H = 0.6 in.	N_{xy}	+0.20			t_{wy}	0.010	0.2358	b_x	LBY	0.0053		
	aluminum					W		16.858		LBP	0.7010		
(1-3)	a = 40 in.	N_x	−0.30			t_s	0.005	0.0448	0.80	GBF	0.9961[a]		
	b = 30 in.	N_y	−0.40			b_x	2.00	2.7709	6.00	LBX	0.0723		
	H = 0.8 in.	N_{xy}	+0.20			t_{wy}	0.010	0.0848	b_x	LBY	0.0964		
	aluminum					W		10.942		LBP	0.9969[a]		
(3-2)	a = 70 in.	N_x	−0.40	+0.50	−1.00	t_s	0.005	0.0520	0.50	GBF	0.8432	0.9754	1.0000[a]
	b = 50 in.	N_y	−0.30	−0.60	0.00	b_x	2.00	3.7051	6.00	LBX	0.0000	−0.0002	0.0003
	H = 0.5 in.	N_{xy}	0.00	−1.00	−0.40	t_{wy}	0.010	0.6421	b_x	LBY	0.0000	+0.0002	0.0000
	steel					W		187.195		LBP	0.1500	+0.0963	0.2614
(3-3)	a = 70 in.	N_x	−0.30	+0.50	−1.00	t_s	0.005	0.3447	0.50	GBF	0.8408	0.9701	0.9996[a]
	b = 50 in.	N_y	−0.30	−0.60	0.00	b_x	2.00	4.0717	6.00	LBX	0.0348	−0.0579	0.1159
	H = 0.5 in.	N_{xy}	0.00	−1.00	−0.40	t_{wy}	0.010	0.0099[a]	b_x	LBY	0.0348	0.0695	0.0000
	titanium					W		193.465		LBP	0.0042	0.0007	0.0070

[a] Indicates a bound design parameter or behavior function.

and local sheet buckling constraints are active in this case [GBF (D_p) = 0.9961, LBP (D_p) = 0.9969]. It is clear that the total depth available has a marked influence on the characteristics of the minimum weight optimum design.

In Ref. 25, it is shown that two relative minima pockets exist for case (1-1). One of these pockets contains the minimum weight thick sheet design and the other contains the minimum weight thin sheet design. Using a program which seeks only the local minimum rather than the global minimum, the two designs shown in Table 3 were obtained (see Ref. 25). In Table 4 a complete synthesis path for case (1-1) using the phase I program is presented and it is apparent that during the synthesis process several stiffened thin sheet designs are obtained. Nevertheless the phase I synthesis program is capable of automatically finding its way back to the thick sheet pocket where it seeks out the minimum weight design. In case (1-2) and (1-3) it is found that the minimum weight optimum design lies in the thin sheet region. This may be attributed to the fact that the assigned total depth H is greater in cases (1-2) and (1-3) than in case (1-1).

Cases (3-2) and (3-3) are three load condition cases using steel and titanium alloy, respectively, as the plate material. These two cases are identical except for the preassigned difference in material. In case (3-2) it is found that the minimum weight design is characterized by a thin sheet and relatively thick stiffener. The only active constraint for the optimum design in case (3-2) is the gross buckling behavior function in load condition three [GBF (D_p) = 1.0000]. Note, however, that the optimum design does lie close to the gross buckling constraint surface of load condition two [GBF (D_p) = 0.9754]. Turning attention to case (3-3), it is found that changing the material to titanium alloy, while holding everything else the same, results in a thick sheet optimum design. In case (3-3) the stiffener thickness is at its lower bound value (t_{wy} = 0.0099 in.) and only the gross buckling behavior function in load condition three [GBF (D_p) =

Table 2 Material properties

Material	Property			
	E, ksi	Y, ksi	μ	ρ, lb/in³
Aluminum	10.5 × 10³	72.0	0.32	0.101
Titanium	16.0 × 10³	120.0	0.29	0.160
Steel	30.0 × 10³	150.0	0.283	0.276

Table 3 Relative minima designs

	Thick sheet minimum	Thin sheet minimum
t_s	0.2803	0.0838
b_x	4.3734	5.2514
t_w	0.0101	2.2969
W	34.0346	36.3488

Table 4 Synthesis path, case 1-1

Cycle	t_s	b_x	t_w	W
1	0.3000	5.0000	4.0000	47.9952
2	0.0334	5.0872	3.8886	46.0138
3	0.1722	5.3338	3.7397	46.0138
4	0.0348	5.3679	3.6905	44.1581
5	0.1149	5.6081	3.6247	44.1581
6	0.0361	5.6243	3.5995	42.7630
7	0.2142	4.9150	2.4388	42.7630
8	0.0531	4.9316	2.4051	37.4435
9	0.1221	5.0286	2.1502	37.4435
10	0.0997	5.0307	2.1453	36.4125
11	0.0978	5.2417	2.2331	36.4125
12	0.0782	5.1701	2.2943	36.4125
13	0.0799	5.1702	2.2942	36.4004
14	0.0929	5.2392	2.2547	36.4004
15	0.0919	5.2393	2.2545	36.3626
16	0.2768	5.2711	0.5234	36.3626
17	0.2669	5.2712	0.5229	35.3855
18	0.2794	3.3337	0.1785	35.3855
19	0.2736	3.3337	0.1781	34.7558
20	0.2802	3.7624	0.1045	34.7558
21	0.2771	3.7624	0.1043	34.3970
22	0.2804	4.2093	0.0595	34.3970
23	0.2788	4.2093	0.0594	34.2034
24	0.2805	4.3460	0.0310	34.2034
25	0.2796	4.3460	0.0310	34.1006
26	0.2805	4.3711	0.0156	34.1006
27	0.2801	4.3711	0.0156	34.0535
28	0.2803	4.3735	0.0117	34.0535
29	0.2802	4.3735	0.0117	34.0346
30	0.2803	4.3735	0.0101	34.0346

Table 5 Synthesis path, case 3-3

Cycle	t_s	b_x	t_w	W
1	0.4000	5.0000	4.0000	277.7600
2	0.0500	5.1283	3.8343	263.9557
3	0.2113	5.3778	3.6836	263.9557
4	0.0519	5.4227	3.6171	252.1764
5	0.1384	5.6508	3.5546	252.1764
6	0.0534	5.6709	3.5224	244.1024
7	0.2675	4.9569	2.3540	244.1024
8	0.0525	4.9825	2.2994	207.3220
9	0.1377	5.0812	2.0402	207.3220
10	0.0911	5.0858	2.0285	197.2403
11	0.1097	5.1611	1.9854	197.2403
12	0.1094	5.1612	1.9853	197.1718
13	0.0980	5.1603	2.0302	197.1718
14	0.0971	5.1604	2.0300	196.9714
15	0.1018	5.2467	2.0453	196.9714
16	0.3383	4.1035	0.1746	196.9714
17	0.3374	4.1035	0.1746	196.5287
18	0.3442	4.1355	0.0906	196.5287
19	0.3413	4.1355	0.0903	194.9753
20	0.3447	4.0262	0.0449	194.9753
21	0.3432	4.0262	0.0447	194.1170
22	0.3449	4.0520	0.0227	194.1170
23	0.3441	4.0520	0.0226	193.6832
24	0.3450	4.0716	0.0114	193.6832
25	0.3446	4.0716	0.0114	193.4650
26	0.3447	4.0717	0.0099	193.4650

0.9996] is active, although it is apparent that the optimum design does lie close to the gross buckling constraint surface of load condition two [GBF (D_p) = 0.9701]. A synthesis path for case (3-3), shown in Table 5, again illustrates the ability of the phase I program to leave the thin sheet design region when a lower weight design can be found in the thick sheet design region.

More detailed phase I information on such things as initial trial designs and multiple synthesis path information will be found in Refs. 25 and 26.

Typical running times for a single synthesis path using the phase I program range from 30–60 min on the Burroughs 220 Digital Computer.§§ Confidence in the proposed optimum designs was developed by a sustained search from the terminal design point. Although convergence was not always completed, running a second synthesis path served to increase confidence in the proposed optimum designs presented.

Table 6 contains a summary of results obtained using the phase II waffle plate synthesis program. Case $(1\text{-}3)_{\text{II}}$ is essentially the same as case (1-3) except that H is not a preassigned parameter and the stiffening need not be symmetric. It is found that the optimum design is characterized by a thin sheet with relatively thin stiffeners. The upper bound on the total depth [$(H)_{\max}$ = 0.8 in.] is not found to be an active side constraint, but the gross buckling, local sheet buckling, and local x stiffener buckling are all active behavior constraints. Note that the stiffening is predominantly parallel to the y axis, that is across the short dimension of the plate. The minimum weight achieved in case $(1\text{-}3)_{\text{II}}$ using the phase II program is $W = 9.5473$ which is 14% less than the minimum weight achieved in case (1-3) using the phase I program. It is reasonable to expect greater weight savings for cases with highly unsymmetric load conditions.

Case $(3\text{-}2)_{\text{II}}$ is essentially the same as case (3-2) except that H is not a preassigned parameter and the stiffening need not be symmetric. It is found that the optimum design is characterized by a thin sheet with relatively thick stiffeners. The upper bound on the total depth [$(H)_{\max}$ = 0.5] is seen to be an active side constraint while the gross buckling behavior constraint in load condition three is also active [GBF (D_p) = 1.0000]. The optimum design achieved in case $(3\text{-}2)_{\text{II}}$ has a weight of 185.75 lb, which is less than 1% below the minimum weight achieved in case (3-2) using the phase I program. This is not surprising when the three load conditions which make up the loading system for case (3-2) and $(3\text{-}2)_{\text{II}}$ are examined.

In running two synthesis paths for case $(3\text{-}2)_{\text{II}}$ it was found that both paths reached a minimum weight of W = 185.75, but the terminal design points for paths A and B, which are shown in Table 7, appear to be distinct design points. Examination of the ratios (t_{wx}/b_y) and (t_{wy}/b_x) for the terminal values given for paths A and B yield: for path A, $(t_{wx}/b_y) = 0.170$ and $(t_{wy}/b_x) = 0.181$; for path B, (t_{wx}/b_y) = 0.170 and $(t_{wy}/b_x) = 0.181$. Since the gross buckling behavior function is the primary active constraint, any design with $(t_{wx}/b_y) = 0.170$, $(t_{wy}/b_x) = 0.181$, $t_s = 0.0475$, and H = 0.5000 will be a minimum weight optimum design if the actual values of t_{wx}, t_{wy}, b_x, and b_y fall within the side constraint bounds and none of the other behavior function constraints are violated.

Case $(3\text{-}2)_{\text{II}}'$ is the same as case $(3\text{-}2)_{\text{II}}$ except that the upper bound on H is raised to $(H)_{\max}$ = 2.5 in. It is found that the optimum design in this case is characterized by a thin sheet and relatively thin stiffeners. It is significant that the terminal value of the total depth H = 0.9866 in. is substantially less than the maximum permitted. The lower bound on b_x is seen to be an active side constraint. The active behavior functions are seen to be gross buckling in load condition three and local sheet buckling in load condition three. The optimum design point is seen to lie close to the y stiffener buckling constraint hypersurface in load condition two [LBY (D_p) = 0.9172]. It is seen that doubling the total depth available has reduced the weight of the optimum design to approximately $\frac{1}{3}$ of the previous weight [case $(3\text{-}2)_{\text{II}}$ W = 185.75, case $(3\text{-}2)_{\text{II}}'$ W = 68.5]. However, increasing the depth available above $(H)_{\max}$ = 1 in. does not yield a further weight reduction for the particular problem considered in case $(3\text{-}2)_{\text{II}}$.

More detailed information on initial designs and double synthesis paths will be bound in Ref. 27. Typical running times for a single synthesis path using the phase II program range from 4–8 hr on the Burroughs 220 Digital Computer.‖‖ Doubling the number of design parameters appears to have increased the effort required to seek out the optimum by a factor of eight.

Conclusions

Based on the analysis presented in Appendix A, automated minimum weight optimum design capabilities have been developed for waffle plates with integral orthogonal stiffeners. In the waffle-plate structural system, the design parameters describe the actual configuration rather than an idealized structure, and instability failure modes predominate as the limiting conditions on the minimum weight optimum design. The development of a structural synthesis capability for waffle plates represents a considerable advance beyond the exploratory truss studies and adds to the growing body of evidence supporting the contention that a structural synthesis capability can be developed for complex structural systems of practical importance.

Future structural synthesis research should seek to bring an ever more meaningful class of structural design problems within the grasp of automated optimization. Further study of specific elementary systems will be useful. Such studies will provide a vehicle for seeking to optimize the method of optimization, and also permit exploration of the interrelation between design philosophies, synthesis, and analysis.

The development of synthesis capabilities for certain general problem classes is thought to be worthwhile. For ex-

§§ It is estimated that running times on an IBM 7090 would range from 1–2 min.

‖‖ It is estimated that running times on an IBM 7090 would range from 8–16 min.

Table 6 Summary of phase II results

Case	Preassigned parameters	Load conditions, N	1	2	3	Design parameters	$D_p^{(L)}$	D_p	$D_p^{(U)}$	Behavior functions, $BF(D_p)$	1	2	3
$(1\text{-}3)_{II}$	a = 40 in.	N_x	−0.30			t_s	0.005	0.0355	H	GY	0.1609		
	b = 30 in.	N_y	−0.40			H	t_s	0.7827	0.800	SX	0.0922		
		N_{xy}	+0.20			b_x	2.00	2.0001[a]	8.00	SY	0.0799		
						b_y	2.0	2.0002[a]	6.00	GBF	0.9998[a]		
						t_{wx}	0.010	0.0260	b_y	LBX	0.9874[a]		
						t_{wy}	0.010	0.0911	b_x	LBY	0.0709		
	aluminum					W		9.5473		LBP	1.0000[a]		
$(3\text{-}2)_{II}$	a = 70 in.	N_x	−0.30	+0.50	−1.00	t_s	0.005	0.0475	H	GY	0.0158	0.2484	0.1112
	b = 50 in.	N_y	−0.30	−0.60	0.00	H	t_s	0.5000[a]	0.500	SX	0.0161	0.0268	0.0537
		N_{xy}	0	−1.00	−0.40	b_x	2.00	5.2688	14.00	SY	0.0154	0.0309	0.0000
						b_y	2.00	3.8928	10.00	GBF	0.8426	0.6407	1.0000
						t_{wx}	0.010	0.6604	b_y	LBX	0.0001	−0.0002	0.0003
						t_{wy}	0.010	0.9544	b_x	LBY	0.0000	0.0001	0.0000
	steel					W		185.75		LBP	0.2607	0.3687	0.3580
$(3\text{-}2)_{II}'$	a = 70 in.	N_x	−0.30	+0.50	−1.00	t_s	0.005	0.0432	H	GY	0.0350	0.2894	0.1570
	b = 50 in.	N_y	−0.30	−0.60	0.00	H	t_s	0.9866	2.50	SX	0.0345	0.0576	0.1151
		N_{xy}	0.00	−1.00	−0.40	b_x	2.00	2.0504[a]	14.00	SY	0.0354	0.0709	0.0000
						b_y	2.00	4.6626	10.00	GBF	0.8262	0.6167	0.9783[a]
						t_{wx}	0.010	0.0726	b_y	LBX	0.0507	−0.0846	0.1693
						t_{wy}	0.010	0.0287	b_x	LBY	0.4586	0.9172	0.0000
	steel					W		68.499		LBP	0.3551	0.1971	1.0000[a]

[a] Indicates a bound design parameter or behavior function.

ample, the design of minimum weight structural systems to perform satisfactorily, when subjected to a multiplicity of static loading conditions, constitutes a problem class for which a synthesis capability is thought to be attainable. The envisioned capability would be able to handle a sizeable number of configuration and sizing design parameters as well as a substantial number of distinct load conditions. There exists a variety of general, highly organized, discrete element methods of structural analysis that are in widespread use, and that provide a technology for developing a general static structural synthesis capability.

To achieve synthesis capabilities of practical proportions will require improved methods of synthesis and the use of the largest and highest speed computers available. However, the authors believe that some form of generalized structural synthesis capability will soon be as commonplace as the highly organized discrete element matrix methods of structural analysis are today.

Appendix A: Governing Technology of Waffle Plates

The first step in developing a procedure for minimum weight balanced design of integrally stiffened waffle plates is to put together a method of analysis that adequately predicts the behavior of such plates. The governing technology to be used to develop a synthesis capability for integrally stiffened waffle plates is given in this appendix. Although no detailed derivations are presented, the specialization of expressions to apply to the synthesis of orthogonally stiffened waffle plates is outlined. A discussion of the assumptions and restrictions imposed by the analysis is incorporated into the presentation of the analysis.

Orthotropic Plate Equations

Many researchers have studied individual characteristics of integrally stiffened flat plates, cylinders and curved panels. Dow, Libove, and Hubka[28] derive formulas for the elastic constants of the equivalent orthotropic plate. This analysis facilitates the use of orthotropic plate theory for the class of flat structures known as integrally stiffened waffle plates. Since the investigation is restricted to flat rectangular waffle plates with simply supported edges, subject to any combination of inplane loads N_x, N_y, and N_{xy}, the governing differential equation is

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{(A1)}$$

The gross instability of the equivalent plate is studied employing a linear elastic buckling technology and the assumed mode technique, and hence can be viewed as an eigenvalue problem.

Gross Buckling of a Waffle Plate

The gross buckling behavior of the waffle plate is studied by first transforming the waffle plate to its equivalent orthotropic plate via the elastic constants and then employing a linear elastic analysis to determine the buckling loads. The assumed mode used throughout is

$$w(x,y) = \sum_n \sum_m A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad \text{(A2)}$$

which was reduced from the general Fourier series. For the case of biaxial compression with no shear load the expression

$$\left[\frac{N_x}{(N_x)_{cr}}\right] + \left[\frac{N_y}{(N_y)_{cr}}\right] = 1$$

gives an exact solution to the orthotropic plate equation if

Table 7 Case $(3\text{-}2)_{II}$

	Path A terminal values	Point B terminal values
t_s	0.0475	0.0475
H	0.5000	0.5000
b_x	5.2688	2.0077
b_y	3.8928	5.1094
t_{wx}	0.6604	0.8678
t_{wy}	0.9544	0.3634
W	185.75	185.75

both $(N_x)_{cr}$ and $(N_y)_{cr}$ are required to be of the same buckling mode.

To find $(N_x)_{cr}$ and $(N_y)_{cr}$ the forementioned expression is written in the following forms:

$$N^x = \frac{(N_x)_r}{1 + \beta\alpha_r^2(n/m)^2}$$

$$N^y = \frac{(N_y)_r}{1 + (1/\beta)(b/a)^2(m/n)^2}$$

where $\beta = N_y/N_x$, $(N_x)_r$ and $(N_y)_r$ are given by (A4) and (A5), and $(N_x)_{cr}$ and $(N_y)_{cr}$ are the values of (A4) and (A5) using the values of m and n for the critical mode. The critical values of m and n are obtained by finding their values such that the smallest positive value of N^x is obtained when N_x is compressive or N_y is zero. When N_x is tensile or zero the critical values of m and n are obtained by finding the smallest positive value of N^y.

Lekhnitski[29] gives an interaction expression for a rectangular plate subject to compression and shear. A single interaction expression can be conjectured from these interaction expressions:

$$\left[\frac{N_x}{(N_x)_{cr}} + \frac{N_y}{(N_y)_{cr}}\right] + \left[\frac{N_{xy}}{(N_{xy})_{cr}}\right]^2 = 1 \tag{A3}$$

It is readily apparent that this expression reduces to the accepted interaction expressions for any combination of two inplane loads, applied to flat orthotropic and isotropic plates.

Since the combined interaction expression reduces to all possible sub-cases it may be considered as a proposed interaction formula for flat orthotropic plates subject to any combination of three inplane loads N_x, N_y, and N_{xy}.

Since any synthesis process must be based on an accepted analysis, the composite interaction expression (A3) can currently be used with confidence only for design load systems made up of any combination of two loads. It is intended that the synthesis capability developed be based on Eq. (A3) and apply for design load systems made up of any combination of two loads. At the same time, this synthesis capability, based on the most plausible analysis available, is capable of handling design load systems made up of three inplane forces.

In order to use the interaction expression (A3) it is necessary to have expressions for the critical loads $(N_x)_{cr}$, $(N_y)_{cr}$, and $(N_{xy})_{cr}$. Lekhnitski[29] gives the expression for orthotropic plate buckling under normal load N_x. Notice that the expression for $(N_y)_r$ can also be derived from the expression for $(N_x)_r$ by a simple permutation of subscripts. The expression for $(N_x)_r$ and $(N_y)_r$ are as follows:

$$(N_x)_r = + \frac{\pi^2(D_1D_2)^{1/2}}{b^2}[(D_1/D_2)^{1/2}(m/\alpha_r)^2 + 2D_3n^2/(D_1D_2)^{1/2} + (D_2/D_1)^{1/2}(\alpha_r n^2/m)^2] \tag{A4}$$

$$(N_y)_r = + \frac{\pi^2(D_1D_2)^{1/2}}{a^2}[(D_2/D_1)^{1/2}(n\alpha_r)^2 + 2D_3m^2/(D_1D_2)^{1/2} + (D_1/D_2)^{1/2}(m^2/\alpha_r n)^2] \tag{A5}$$

To find $(N_x)_{cr}$ and $(N_y)_{cr}$ take the negative of $(N_x)_r$ and $(N_y)_r$, respectively, using the critical values of m and n.

Seydel[30] presents a study of the instability of simply supported orthotropic plates under an inplane shear loading. The notation used as a means of simplifying the expressions for shear buckling is $\theta = (D_1D_2/D_3^2)^{1/2}$. The following definitions apply only for $\theta > 1$ which is always true for waffle plates:

$$\beta = (b/a)(D_1/D_2)^{1/4}$$

$$\varphi(m,n) = (m\beta)^4 + 2(m\beta)^2(n^2/\theta) + n^4 \tag{A6}$$

$$(N_{xy})_r = + C_a\frac{(D_1D_2^3)^{1/4}}{(b/2)^2}$$

where m and n are integer parameters of the assumed mode expansion. One additional restriction is that the value of β must always be between zero and unity. Since D_1 and D_2 are functions of the design parameters and can change throughout the synthesis, provision must be made to keep β between zero and unity. This can be done by effectively switching the plate around and solving for N_{yx} and then switching the plate back (note that for equilibrium $N_{xy} = N_{yx}$). The buckling coefficient C_a is given by the following expressions: for $n = 1, 2, 3$, and $m = q, q+1, q+2$.

Case I: Symmetric buckling with q odd or antisymmetric buckling with q even:

$$C_a = \frac{\pi^4}{128}\left\{\frac{[\varphi(q+1,2)]^{1/2}}{2(q+1)\beta}\right\}\left\{\left[\frac{q}{2q+1}\right]^2\left[\frac{1}{9\varphi(q,1)} + \frac{9}{25\varphi(q,3)}\right] + \left[\frac{q+2}{2q+3}\right]^2\left[\frac{1}{9\varphi(q+2,1)} + \frac{9}{25\varphi(q+2,3)}\right]\right\}^{-1/2} \tag{A7}$$

Case II: Symmetric buckling with q even or, antisymmetric buckling with q odd:

$$C_a = \frac{\pi^4}{128}\left\{\frac{1}{2(q+1)\beta}\right\}\left\{\left[\frac{1}{9\varphi(q+1,1)} + \frac{9}{25\varphi(q+1,3)}\right] \times \left[\frac{q^2}{(2q+1)^2\varphi(q,2)} + \frac{(q+2)^2}{(2q+3)^2\varphi(q+2,2)}\right]\right\}^{-1/2} \tag{A8}$$

Notice that there exists a unique expression for C_a for both symmetric and antisymmetric buckling. Stein and Neff[7] showed that either the symmetric or antisymmetric mode can be critical depending upon α_r, the aspect ratio. This is also true for orthotropic plates but the dependency is upon β where $\beta = (1/\alpha_r)(D_1/D_2)^{1/4}$. The critical value of $(N_{xy})_r$ must therefore be determined from two complete sets of mode parameters. That is, given the values of θ and β as data, determine the minimum magnitude of $(N_{xy})_r$ as a function of the integer parameter q and gross buckling pattern (symmetric or antisymmetric).

The forementioned expressions are used in conjunction with the interaction expression (A3) to give the necessary technology to predict the gross buckling behavior of simply supported waffle plates. This analysis constitutes a major portion of the governing technology upon which the synthesis capability is based. Up to this point the analysis has dealt only with orthotropic plates, in general. In order to use the analysis, it is necessary to transform the waffle plate into an equivalent orthotropic plate. The equation of the elastic constants used for this transformation were derived by Dow, Libove, and Hubka.[28]

The gross buckling phenomenon of the waffle plate is studied by generating an equivalent plate and examining the buckling characteristics of this plate. A detailed derivation of the transformation is outlined in Ref. 28. The basic assumptions or restrictions necessary to make this transformation are as follows:

1) The rib spacings of the integrally-stiffened plate are small in comparison with the over-all width and length of the plate. This assumption is made in order that the average or over-all behavior may be studied rather than a detailed study of any particular segment. Several studies have been made where the rib spacing was large (i.e., one, two, or even three ribs in either or both directions). In this case, the behavior cannot be examined on a gross scale, but a detailed elastic stability study of the interacting subsystems has to be performed.

2) This particular analysis is concerned with waffle plates with only longitudinal and transverse ribs. The expressions for the elastic constants are specialized from those given in Ref. 28. It is important to notice that the restriction in no way limits the gross instability analysis. In order to extend

this study to include skewed stiffeners, only a modification of the expressions for the elastic constants would be necessary.

3) Since the shear stress (τ_{xy}) is zero at the outer boundary of each stiffener, it is assumed that the shear stress (τ_{xy}) is zero throughout each stiffener. The total shear load therefore must be carried by the back-up sheet.

4) The formulas for the equivalent elastic constants involve coefficients α, β, α', and β' which define the effectiveness of a rib in resisting transverse stretching and bending, and inplane shearing and twisting. The terms β and β' represent that fraction of the volume of a rib resisting stretching and shearing respectively. The terms α and α' locate the centers of gravity of these effective volumes. The lower bound of zero is assumed for β and β' and consequently the values of α and α' are immaterial.

Consider a waffle plate with only one set of stiffeners subject to a load N_x which is transverse to the stiffeners. It is readily apparent that the normal stress in the sheet midway between two ribs is N_x/t_s if the stiffener spacing is large compared to the stiffener thickness. It is also true that the normal stress in the vicinity of a rib is lower than the normal stress midway between two ribs. However, a uniform stress equal to that at the midpoint is assumed to exist throughout. That is, it is assumed that a rib has no effect on the stress distribution generated by a transverse load.

With the forementioned restrictions, the expressions for the flexural rigidities and torsional rigidity for the symmetric waffle are

$$D_1 = D_2 = EH^3\left\{\frac{1}{12(1-\mu^2)}\left(\frac{t_s}{H}\right)^3 + \frac{1}{12}\left(1-\frac{t_s}{H}\right)^3 \times \left(\frac{t_w}{b_x}\right) + \frac{\mu}{8(1-\mu^2)}\left(1-\frac{t_s}{H}\right)\left(\frac{t_s}{H}\right)\left(\frac{t_w}{b_x}\right) \times \left[\frac{(1+\mu)}{\frac{\mu}{1-\mu}\left(\frac{t_s}{H}\right) + \mu\left(1-\frac{t_s}{H}\right)\left(\frac{t_w}{b_x}\right)} + \frac{(1-\mu)}{\frac{\mu}{1+\mu}\left(\frac{t_s}{H}\right) + \mu\left(1-\frac{t_s}{H}\right)\left(\frac{t_w}{b_x}\right)}\right]\right\} \quad \text{(A9)}$$

and

$$2D_3 = \frac{EH^3}{2}\left\{\frac{1}{3(1-\mu^2)}\left(\frac{t_s}{H}\right)^3 + \frac{\frac{\mu}{(1-\mu^2)}\left(\frac{t_s}{H}\right)\left(1-\frac{t_s}{H}\right)^2\left(\frac{t_w}{b_x}\right)^2}{\left[\frac{1}{(1-\mu^2)}\left(\frac{t_s}{H}\right) + \left(1-\frac{t_s}{H}\right)\left(\frac{t_w}{b_x}\right)\right]^2 - \left(\frac{1}{(1-\mu^2)}\right)^2\left(\frac{t_s}{H}\right)^2}\right\} \quad \text{(A10)}$$

For the waffle that is not restricted to being symmetric, the expressions are slightly more complicated.

$$D_1 = EH^3\left[I_x - \frac{A_s^2 A_x}{\bar{A}_s^2}(\bar{k}_x)^2\right] \quad \text{(A9'a)}$$

$$D_2 = EH^3\left[I_y - \frac{A_s^2 A_y}{\bar{A}_s^2}(\bar{k}_y)^2\right] \quad \text{(A9'b)}$$

$$D_3 = \frac{EH^3}{2}\left[2\frac{\bar{I}_s^2}{\bar{A}_s^2} + I_{xy}\right] \quad \text{(A10')}$$

where

$$I_x = \frac{1}{12(1-\mu^2)}\left(\frac{t_s}{H}\right)^3 + \frac{I_{wx}}{b_yH^3} + \frac{1}{4A_x^2(1-\mu^2)}\left(\frac{t_s}{H}\right)\left(\frac{A_{wx}}{b_yH}\right)^2 + \frac{1}{4}\left(\frac{A_{wx}}{b_yH}\right)\left[1 - \frac{1}{A_y}\left(\frac{A_{wy}}{b_xH}\right)\right]^2$$

$$I_y = \frac{1}{12(1-\mu^2)}\left(\frac{t_s}{H}\right)^3 + \frac{I_{wy}}{b_xH^3} + \frac{1}{4A_y^2(1-\mu)^2}\left(\frac{t_s}{H}\right)\left(\frac{A_{wy}}{b_xH}\right)^2 + \frac{1}{4}\left(\frac{A_{wy}}{b_xH}\right)\left[1 - \frac{1}{A_y}\left(\frac{A_{wy}}{b_xH}\right)\right]^2$$

and

$$\bar{A}_s^2 = A_xA_y - A_s^2$$

$$\bar{k}_x = \frac{1}{2A_x}\left(\frac{A_{wx}}{b_yH}\right)$$

$$\bar{k}_y = \frac{1}{2A_y}\left(\frac{A_{wy}}{b_xH}\right)$$

$$A_x = \frac{A_s}{\mu} + \frac{A_{wx}}{b_yH}$$

$$A_y = \frac{A_s}{\mu} + \frac{A_{wy}}{b_xH}$$

$$A_s = \frac{\mu}{1-\mu^2}\left(\frac{t_s}{H}\right)$$

$$\bar{I}_s^2 = I_s\bar{A}_s^2 + A_sA_xA_y\bar{k}_x\bar{k}_y$$

$$I_{xy} = \frac{1}{6(1+\mu)}\left(\frac{t_s}{H}\right)^3$$

Then within these expressions the following definitions are needed to obtain the equivalent elastic constants in terms of the design parameters:

$$\frac{I_{wx}}{b_yH^3} = \frac{1}{12}\left(1-\frac{t_s}{H}\right)^3\left(\frac{t_{wx}}{b_y}\right)$$

$$\frac{I_{wy}}{b_xH^3} = \frac{1}{12}\left(1-\frac{t_s}{H}\right)^3\left(\frac{t_{wy}}{b_x}\right)$$

$$\frac{A_{wx}}{b_yH} = \left(1-\frac{t_s}{H}\right)\left(\frac{t_{wy}}{b_y}\right)$$

$$\frac{A_{wy}}{b_xH} = \left(1-\frac{t_s}{H}\right)\left(\frac{t_{wy}}{b_x}\right)$$

$$I_s = \frac{\mu}{12(1-\mu^2)}\left(\frac{t_s}{H}\right)^3$$

Local Buckling of a Waffle Plate

Two distinct modes of local buckling of waffle plates are considered. Examine the case where the stiffeners are thin compared to the sheet. Then it is possible for the sheet to undergo a stable inplane displacement while the stiffeners become unstable in a mode comparable to flange buckling. On the other hand, if the sheet is thin compared to the stiffeners, it is possible for each individual panel to buckle as a rectangular plate while the stiffeners undergo a stable inplane deflection. In both cases the mathematical model is a rectangular isotropic plate.

Consider the case of stiffener instability that occurs when the stiffener thickness is small compared to the sheet thickness. The model used for analysis is the portion of a longitudinal stiffener between two adjacent transverse stiffeners. The boundary conditions are taken as follows: hinged ends, hinged on one edge and free on the other edge. The applied load is a uniformly distributed normal load applied along the hinged ends of the plate.

The expressions for the critical loads on the stiffeners of a symmetric waffle are as follows:

$$(\bar{N}_x)_{cr} = -\pi^2 D' \left(\frac{t_w}{H - t_s}\right)^2 \left[\frac{b_x t_s + t_w(H - t_s)}{b_x}\right] K_B \quad \text{(A11)}$$

and

$$(\bar{N}_y)_{cr} = -\pi^2 D' \left(\frac{t_w}{H - t_s}\right)^2 \left[\frac{b_y t_s + t_w(H - t_s)}{b_x}\right] K_B \quad \text{(A12)}$$

where

$$K_B = \left(\frac{1}{\varphi}\right)^2 + 0.425 \quad \text{(A13)}$$

$$\varphi = \frac{b_x - t_w}{H - t_s}$$

$$D' = \frac{E}{12(1 - \mu^2)}$$

and for the waffle that is not restricted to being symmetric, the expressions for the critical loads on the stiffeners are

$$(\bar{N}_x)_{cr} = -\frac{\pi^2 E(t_{wx}^3)}{12(1 - \mu^2)} \left[\frac{t_s}{t_{wx}} + \frac{H - t_s}{b_y}\right] \left[\frac{1}{(H - t_s)^2}\right] \times \left[\frac{(H - t_s)^2}{(b_x - t_{wy})^2} + 0.425\right] \quad \text{(A11')}$$

$$(\bar{N}_y)_{cr} = -\frac{\pi^2 E(t_{wy}^3)}{12(1 - \mu^2)} \left[\frac{t_s}{t_{wy}} + \frac{H - t_s}{b_x}\right] \left[\frac{1}{(H - t_s)^2}\right] \times \left[\frac{(H - t_s)^2}{(b_y - t_{wx})^2} + 0.425\right] \quad \text{(A12')}$$

Consider the case of sheet instability which occurs when the sheet thickness is small compared to the stiffener thickness. The mathematical model used for this system is a rectangular isotropic plate bounded by two transverse stiffeners and two longitudinal stiffeners which are assumed to provide hinged boundary conditions. Since it is possible for the sheet to carry all three inplane loads N_x, N_y, and N_{xy}, the interaction expression

$$\left(\frac{N_x}{(\tilde{N}_x)_{cr}}\right) + \left(\frac{N_y}{(\tilde{N}_y)_{cr}}\right) + \left(\frac{N_{xy}}{(\tilde{N}_{xy})_{cr}}\right)^2 = 1 \quad \text{(A14)}$$

must be used to predict the instability of the back-up sheet.

For the case of the symmetric waffle the expressions for $(\tilde{N}_x)_{cr}$, $(\tilde{N}_y)_{cr}$, and $(\tilde{N}_{xy})_{cr}$ are as follows:

$$(\tilde{N}_x)_{cr} = -4.00 \frac{\pi^2 D}{(b_x - t_w)^2} \left[\frac{t_s b_x + t_w(H - t_s)}{t_s b_x}\right] \quad \text{(A15)}$$

$$(\tilde{N}_y)_{cr} = -4.00 \frac{\pi^2 D}{(b_x - t_w)^2} \left[\frac{t_s b_x + t_w(H - t_s)}{t_s b_x}\right] \quad \text{(A16)}$$

$$(\tilde{N}_{xy})_{cr} = \pm 9.34 \left[\frac{\pi^2 E t_s^3}{12(1 - \mu^2)(b_x - t_w)^2}\right] \quad \text{(A17)}$$

For the waffle that is not restricted to being symmetric, the problem of determining a local instability is complicated by the fact that the buckling mode is not predetermined. One must use an analysis similar to that used in detecting a gross instability with $D_1 = D_2 = D_3 = D$:

$$(\tilde{N}_x)_r = \frac{\pi^2 D}{(b_y - t_{wx})^2} \left[m^2 \left(\frac{b_y - t_{wx}}{b_x - t_{wy}}\right)^2 + 2n^2 + \frac{n^4}{m^2} \left(\frac{b_x - t_{wy}}{b_y - t_{wx}}\right)^2\right] F_x \quad \text{(A15')}$$

$$(\tilde{N}_y)_r = \frac{\pi^2 D}{(b_x - t_{wy})^2} \left[n^2 \left(\frac{b_x - t_{wy}}{b_y - t_{wx}}\right)^2 + 2m^2 + \frac{m^4}{n^2} \left(\frac{b_y - t_{wx}}{b_y - t_{wy}}\right)^2\right] F_y \quad \text{(A16')}$$

and

$$D = \frac{E t_s^3}{12(1 - \mu^2)}$$

The factors F_x and F_y take into account that portion of the load which is carried by the sheet and they are

$$F_x = \left[1 - \left(1 - \frac{H}{t_s}\right)\left(\frac{t_{wx}}{b_y}\right)\right]$$

$$F_y = \left[1 - \left(1 - \frac{H}{t_s}\right)\left(\frac{t_{wy}}{b_x}\right)\right]$$

$$\tilde{\beta} = \tilde{\beta} \frac{F_x}{F_y}$$

It is important that m and n are the same in both expressions and must be used consistantly throughout the local plate buckling analysis.

With simular substitution, the shear buckling equations become

$$\tilde{N}_{xy} = \frac{4C_a D}{\tilde{b}^2} \quad \text{(A17')}$$

where in this case $\tilde{b}$ is the smaller and $\tilde{a}$ is the larger of the values of $(b_x - t_{wy})$ and $(b_y - t_{wx})$. The C_a are the same as before with

$$\beta = \tilde{b}/\tilde{a} \qquad \theta = 1$$

$$\varphi(m,n) = \left[\left(m\frac{\tilde{b}}{\tilde{a}}\right)^2 + n^2\right]^2$$

The foregoing expressions give the equations necessary to predict local waffle plate instability when it occurs as stiffener or sheet buckling. The one remaining mode of failure is the material yield criterion which is presented in the next section.

Material Yield Criterion

Under normal applications, the design of waffle plates is governed by a buckling criterion. Since the synthesis process is developed from a linear theory of elastic buckling, it is necessary to know when the final design is such that the analysis is not valid and an inelastic buckling theory should be employed. A material yield criterion is used as an alarm to determine when the final design is not governed by the elastic buckling constraints. Because of the nature of the synthesis process, it is possible for an intermediate design to be governed by a material yield criterion. If the redesign process continues and finds a new design which is governed by an elastic buckling constraint, the intermediate design is subsequently discarded. In short, the analysis assumes an ideal elastic-plastic behavior of the structural material.

The distortion energy criterion, employed as the material yield alarm, is as follows:

$$\{\alpha_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2\}^{1/2} = Y \quad \text{(A18)}$$

The expression for stresses are substituted into (A18) to give the yield condition for the symmetric waffle:

$$Y = \frac{1}{H}\left\{\frac{(N_x^2 - N_x N_y + N_y^2)}{\left[\left(\frac{t_s}{H}\right) + \left(1 - \frac{t_s}{H}\right)\frac{t_w}{b_x}\right]^2} + 3\left(\frac{N_{xy}}{t_s/H}\right)^2\right\}^{1/2} \quad \text{(A19)}$$

and for the waffle that is not restricted to being symmetric:

$$\left\{\frac{N_x^2 b_y^2}{[b_y t_s + t_{wx}(H - t_s)]^2} - \frac{N_x N_y b_y b_x}{[b_y t_s + t_{wx}(H - t_s)][b_x t_s + t_{wy}(H - t_s)]} + \frac{N_y^2 b_x^2}{[b_x t_s + t_{wy}(H - t_s)]^2} + 3\left(\frac{N_{xy}}{t_s}\right)^2\right\}^{1/2} = Y \quad \text{(A19')}$$

The ordering of principal stresses is unnecessary and the foregoing expressions are valid regardless of the magnitude or sign of the applied loads.

A uniaxial state of stress exists in each stiffener and the distortion energy criterion reduces to

$$Y = \sigma_{\max} - \sigma_{\min}$$

Since the coefficient relating stress to load is always positive, the stiffener yield criteria can be expressed as

$$Y = |N_x|\left[\frac{b_y}{t_s b_y + t_{wx}(H - t_s)}\right] \quad \text{(A20) and (A20')}$$

$$Y = |N_y|\left[\frac{b_x}{t_s b_y + t_{wy}(H - t_s)}\right] \quad \text{(A21) and (A21')}$$

for both symmetric and unsymmetric configurations of stiffeners.

Appendix B: Phase I—Alternate Step Points

In solving for the direction of travel in the alternate step mode it is necessary to know the coordinates of the intersection of the random plane, the tangent plane, and the side constraint plane as well as the coordinates of the intersection of the random plane, current weight surface, and the constraint plane. Since there are three types of random planes and it is possible to move in two distinct directions in each random plane, there exists twelve sets of coordinates and they are:

Type I

Point 1: Solve the following simultaneously:

$$b_x = (b_x)_{\max}$$

$$W^{(q)} = \rho a\, bH\left[1 - \left(1 - \frac{t_s}{H}\right)\left(1 - \frac{t_{wy}}{b_x}\right)^2\right]$$

$$Bb_x + Ct_{wy} = D_c$$

to get the coordinates

$$b_{x_1} = (b_x)_{\max}$$

$$t_{wy_1} = \frac{D_c - Bb_{x_1}}{C}$$

$$t_{s_1} = H\left[1 - \frac{1 - (W^{(q)}/ab\rho H)}{[1 - (t_{wy_1}/b_{x_1})]^2}\right]$$

Point 2: Solve the following simultaneously:

$$b_x = (b_x)_{\max}$$

$$\left(\frac{\partial W}{\partial t_s}\right)_c t_s + \left(\frac{\partial W}{\partial b_x}\right)_c b_x + \left(\frac{\partial W}{\partial t_{wy}}\right)_c t_{wy} = K_c$$

$$Bb_x + Ct_{wy} = D_c$$

to get the coordinates

$$b_{x_2} = (b_x)_{\max}$$

$$t_{wy_2} = \frac{D_c - Bb_{x_2}}{C}$$

$$t_{s_2} = \frac{K_c - \left(\frac{\partial W}{\partial b_x}\right)_c b_{x_2} - \left(\frac{\partial W}{\partial t_{wy}}\right)_c t_{wy_2}}{\left(\frac{\partial W}{\partial t_s}\right)_c}$$

Point 3: Solve the following simultaneously:

$$t_s = (t_s)_{\min} \approx 0$$

$$W^{(q)} = \rho a\, bH\left[1 - \left(1 - \frac{t_{wy}}{b_x}\right)^2\right]$$

$$Bb_x + Ct_{wy} = D_c$$

to get the coordinates

$$t_{s_3} = 0$$

$$b_{x_3} = \frac{D_c}{B + \left[1 - \left(1 - \frac{W^{(q)}}{ab\rho H}\right)^{1/2}\right]C}$$

$$t_{wy_3} = \frac{D_c\left[1 - \left(1 - \frac{W^{(q)}}{ab\rho H}\right)^{1/2}\right]}{B + \left[1 - \left(1 - \frac{W^{(q)}}{ab\rho H}\right)^{1/2}\right]} = b_{x3}\left[1 - \left(1 - \frac{W^{(q)}}{ab\rho H}\right)^{1/2}\right]$$

Point 4: Solve the following simultaneously:

$$t_s = (t_s)_{\min} \approx 0$$

$$\left(\frac{\partial W}{\partial t_s}\right)_c t_s + \left(\frac{\partial W}{\partial b_x}\right)_c b_x + \left(\frac{\partial W}{\partial t_{wy}}\right)_c t_{wy} = K_c$$

$$Bb_x + Ct_{wy} = D_c$$

to get the coordinates

$$t_{s_4} = 0$$

$$t_{wy4} = \frac{\left(\frac{\partial W}{\partial b_x}\right)_c D_c - BK_c}{\left(\frac{\partial W}{\partial b_x}\right)_c C - \left(\frac{\partial W}{\partial t_{wy}}\right)_c B}$$

$$b_{x_4} = \frac{1}{B}[D_c - Ct_{wy_4}]$$

Type II

Point 1: Solve the following simultaneously:

$$t_{wy} = (t_{wy})_{\min} \approx 0$$

$$Bb_x + Ct_{wy} = D_c$$

$$W^{(q)} = ab\rho H\left[1 - \left(1 - \frac{t_s}{H}\right)\right]$$

to get the coordinates

$$t_{wy_1} = 0$$

$$b_{x_1} = \frac{D_c}{B}$$

$$t_{s_1} = \frac{W^{(q)}}{ab\rho}$$

Point 2: Solve the following simultaneously:

$$t_{wy} = (t_{wy})_{\min} \approx 0$$

$$Bb_x + Ct_{wy} = D_c$$

$$\left(\frac{\partial W}{\partial t_s}\right)_c t_s + \left(\frac{\partial W}{\partial b_x}\right)_c b_x + \left(\frac{\partial W}{\partial t_{wy}}\right)_c t_{wy} = K_c$$

to get the coordinates

$$t_{wy_2} = 0$$

$$b_{x2} = \frac{D_c}{B}$$

$$t_{s_2} = \frac{1}{\left(\frac{\partial W}{\partial t_s}\right)_c} \left[K_c - \left(\frac{\partial W}{\partial b_x}\right)_c b_{x_2} \right]$$

Points 3 and 4 are identical with points 3 and 4 of type I.

Type III

Points 1 and 2 are identical with points 1 and 2 of type II. Points 3 and 4 are identical with points 1 and 2 of type I.

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